Introduction to Logic (CS & MA) 2015

Answers Exam of 5 November

1. (a) $\neg A \to (\neg R \to C)$ [or: $\neg A \to (R \vee C)$]
   A: Aida is away.
   C: Aida will come.
   R: It rains.

   (b) $(\neg F \wedge \neg A) \wedge (L \to H)$
   F: Serge feels ill.
   A: Serge avoids social contacts.
   L: Serge leaves his room.
   H: Serge is hungry.

2. (a) $\exists x \exists y(\neg(x = y) \wedge m(x) = v \wedge m(y) = v \wedge A(x, v) \wedge A(y, v))$

   (b) $\forall x(m(x) = j \to (x = m(v) \vee A(x, v)))$

   (c) $\neg \exists x \forall y(A(y, m(y)) \to A(x, y))$

3. (a)

A	B	C	$\neg A$	$\leftrightarrow$	$(\neg B$	$\to$	$C)$	$\Leftrightarrow$	$(\neg A$	$\wedge$	$(\neg C$	$\to$	$B))$	$\vee$	$((A$	$\wedge$	$\neg C)$	$\wedge$	$\neg B)$
T	T	T	F	F	F	T	T	T	F	F	F	T	T	F	T	F	F	F	F
T	T	F	F	F	F	T	F	T	F	F	T	T	T	F	T	T	T	F	F
T	F	T	F	F	T	T	T	T	F	F	F	T	F	F	T	F	F	F	T
T	F	F	F	T	T	F	F	T	F	F	T	F	F	T	T	T	T	T	T
F	T	T	T	T	F	T	T	T	T	T	F	T	T	T	F	F	F	F	F
F	T	F	T	T	F	T	F	T	T	T	T	T	T	T	F	F	T	F	F
F	F	T	T	T	T	T	T	T	T	T	F	T	F	T	F	F	F	F	T
F	F	F	T	F	T	F	F	T	T	F	T	F	F	F	F	F	T	F	T
1	2	3	4	10	5	7	3	14	4	11	6	8	2	13	1	9	6	12	5

   The numbers in the last row indicate the order in which the columns are computed. The final column (numbered 14) only contains the value T, so the two sentences are *tautologically equivalent.*

   (b)

A	B	((((	A	$\to$	$B)$	$\to$	$A)$	$\to$	$A)$	$\to$	$B)$	$\to$	A
T	T		T	T	T	T	T	T	T	T	T	T	T
T	F		T	F	F	T	T	T	T	F	F	T	T
F	T		F	T	T	F	F	T	F	T	T	F	F
F	F		F	T	F	F	F	T	F	F	F	T	F
1	2		1	3	2	4	1	5	1	6	2	7	1

   The numbers in the last row indicate the order in which the columns are computed. The final column (numbered 7) does not only contain the value T, so the sentence is *not a tautology.*

4. (a)

1. $(A \to C) \vee (B \to C)$	
2. $\neg C$	
3. $A \wedge B$	
4. A	$\wedge$ Elim: 3
5. B	$\wedge$ Elim: 3
6. $A \to C$	
7. C	$\to$ Elim: 4, 6
8. $B \to C$	
9. C	$\to$ Elim: 5, 8
10. C	$\vee$ Elim: 1, 6–7, 8–9
11. $\bot$	$\bot$ Intro: 10, 2
12. $\neg(A \wedge B)$	$\neg$ Intro: 3–11
13. $\neg C \to \neg(A \wedge B)$	$\to$ Intro: 2–12

(b)

1. $\exists x(A(x) \to B(x))$	
2. $\forall x(A(x) \wedge \neg B(x))$	
3. $\boxed{a}$ $A(a) \to B(a)$	
4. $A(a) \wedge \neg B(a)$	$\forall$ Elim: 2
5. $A(a)$	$\wedge$ Elim: 4
6. $B(a)$	$\to$ Elim: 5, 3
7. $\neg B(a)$	$\wedge$ Elim: 4
8. $\bot$	$\bot$ Intro: 6, 7
9. $\bot$	$\exists$ Elim: 1, 3–8
10. $\neg\forall x(A(x) \wedge \neg B(x))$	$\neg$ Intro: 2–9

(c)

1. $\forall x(a = x \vee b = x)$	
2. $A(a) \wedge A(b)$	
3. $\boxed{c}$	
4. $a = c \vee b = c$	$\forall$ Elim: 1
5. $a = c$	
6. $A(a)$	$\wedge$ Elim: 2
7. $A(c)$	$=$ Elim: 6, 5
8. $b = c$	
9. $A(b)$	$\wedge$ Elim: 2
10. $A(c)$	$=$ Elim: 9, 8
11. $A(c)$	$\vee$ Elim: 4, 5–7, 8–10
12. $\forall y A(y)$	$\forall$ Intro: 3–11
13. $(A(a) \wedge A(b)) \to \forall y A(y)$	$\to$ Intro: 2–12

(d)

1. $\neg\forall x A(x)$	
2. $\neg\exists x \neg A(x)$	
3. $\boxed{a}$	
4. $\neg A(a)$	
5. $\exists x \neg A(x)$	$\exists$ Intro: 4
6. $\bot$	$\bot$ Intro: 5, 2
7. $\neg\neg A(a)$	$\neg$ Intro: 4–6
8. $A(a)$	$\neg$ Elim: 7
9. $\forall x A(x)$	$\forall$ Intro: 3–8
10. $\bot$	$\bot$ Intro: 9, 1
11. $\neg\neg\exists x \neg A(x)$	$\neg$ Intro: 2–10
12. $\exists x \neg A(x)$	$\neg$ Elim: 11

5. (a) $\exists \mathsf{x}(\mathsf{Cube}(\mathsf{x}) \wedge \forall \mathsf{y}(\mathsf{Cube}(\mathsf{y}) \to \mathsf{x} = \mathsf{y}))$

(b) The only object y that satisfies $\neg\exists \mathsf{z}\ \mathsf{LeftOf}(\mathsf{z}, \mathsf{y})$ is the small dodecaeder b. So $\exists \mathsf{y}(\mathsf{SameSize}(\mathsf{x}, \mathsf{y}) \wedge \neg\exists \mathsf{z}\ \mathsf{LeftOf}(\mathsf{z}, \mathsf{y}))$ only holds for objects x that are small, and not for medium object c, nor for the large objects a and d. As a consequence, the sentence is false.

(c) We shall make the sentence true by making the premiss $\exists \mathsf{y}(\mathsf{SameCol}(\mathsf{x}, \mathsf{y}) \wedge \mathsf{x} \neq \mathsf{y})$ false for all objects x. In the present world, this premiss is true only for x equals d or e. When we take away either d or e, the premiss is no longer true for any x, and the full sentence becomes true.

6. (a) $\mathfrak{M} \models P(a) \to (R(y,x) \vee Q(b))[h]$

$\Leftrightarrow$ { definition of satisfaction for implication }

not $\mathfrak{M} \models P(a)[h]$ or $\mathfrak{M} \models (R(y,x) \vee Q(b))[h]$

$\Leftrightarrow$ { definition of satisfaction for disjunction }

not $\mathfrak{M} \models P(a)[h]$ or $\mathfrak{M} \models R(y,x)[h]$ or $\mathfrak{M} \models Q(b)[h]$

$\Leftrightarrow$ { definition of satisfaction for atomic formulae }

$[\![a]\!]_h^{\mathfrak{M}} \notin \mathfrak{M}(P)$ or $\langle [\![y]\!]_h^{\mathfrak{M}}, [\![x]\!]_h^{\mathfrak{M}} \rangle \in \mathfrak{M}(R)$ or $[\![b]\!]_h^{\mathfrak{M}} \in \mathfrak{M}(Q)$

$\Leftrightarrow$ { interpretation of terms }

$\mathfrak{M}(a) \notin \mathfrak{M}(P)$ or $\langle h(y), h(x) \rangle \in \mathfrak{M}(R)$ or $\mathfrak{M}(b) \in \mathfrak{M}(Q)$

$\Leftrightarrow$ { definition of h and $\mathfrak{M}$ }

$1 \notin \{1,3\}$ or $\langle 3,2 \rangle \in \{\langle 1,3\rangle, \langle 2,2\rangle, \langle 2,1\rangle, \langle 3,3\rangle\}$ or $3 \in \{2,3\}$

$\Leftrightarrow$

false or false or true

$\Leftrightarrow$

true

(b) $\mathfrak{M} \models \forall x(P(x) \to R(z,x))[h]$

$\Leftrightarrow$ { definition of satisfaction for universal quantification }

$\mathfrak{M} \models (P(x) \to R(z,x))[h[x/d]]$ for all $d \in \{1,2,3\}$

We claim that this is false. To show this, it suffices to demonstrate that $\mathfrak{M} \models (P(x) \to R(z,x))[h[x/d]]$ is false for some $d \in \{1,2,3\}$. We shall do this for $d = 3$. We use k to abbreviate $h[x/3]$, so $k(x) = 3$ and $k(z) = h(z) = 2$. Now

$\mathfrak{M} \models (P(x) \to R(z,x))[k]$

$\Leftrightarrow$ { definition of satisfaction for implication }

not $\mathfrak{M} \models P(x)[k]$ or $\mathfrak{M} \models R(z,x)[k]$

$\Leftrightarrow$ { definition of satisfaction for atomic formulae }

$[\![x]\!]_k^{\mathfrak{M}} \notin \mathfrak{M}(P)$ or $\langle [\![z]\!]_k^{\mathfrak{M}}, [\![x]\!]_k^{\mathfrak{M}} \rangle \in \mathfrak{M}(R)$

$\Leftrightarrow$ { interpretation of terms }

$k(x) \notin \mathfrak{M}(P)$ or $\langle k(z), k(x) \rangle \in \mathfrak{M}(R)$

$\Leftrightarrow$ { definition of k and $\mathfrak{M}$ }

$3 \notin \{1,3\}$ or $\langle 2,3 \rangle \in \{\langle 1,3\rangle, \langle 2,2\rangle, \langle 2,1\rangle, \langle 3,3\rangle\}$

$\Leftrightarrow$

false or false

$\Leftrightarrow$

false

(c) $\mathfrak{M} \models \forall x \exists y R(x,y)[h]$

$\Leftrightarrow$ { definition of satisfaction for universal quantification }

for all $d \in \{1,2,3\}$: $\mathfrak{M} \models \exists y R(x,y)[h[x/d]]$

$\Leftrightarrow$ { definition of satisfaction for existential quantification }

for all $d \in \{1,2,3\}$ there is an $e \in \{1,2,3\}$ such that $\mathfrak{M} \models R(x,y)[h[x/d, y/e]]$

$\Leftrightarrow$ { definition of satisfaction for atomic sentences }

for all $d \in \{1,2,3\}$ there is an $e \in \{1,2,3\}$ such that $\langle [\![x]\!]^{\mathfrak{M}}_{h[x/d,y/e]}, [\![y]\!]^{\mathfrak{M}}_{h[x/d,y/e]} \rangle \in \mathfrak{M}(R)$

$\Leftrightarrow$ { interpretation of terms }

for all $d \in \{1,2,3\}$ there is an $e \in \{1,2,3\}$ such that $\langle h[x/d, y/e](x), h[x/d, y/e](y) \rangle \in \mathfrak{M}(R)$

$\Leftrightarrow$

for all $d \in \{1,2,3\}$ there is an $e \in \{1,2,3\}$ such that $\langle d, e \rangle \in \mathfrak{M}(R)$

This last statement is true, for we have $\langle 1,3\rangle, \langle 2,2\rangle, \langle 3,3\rangle \in \{\langle 1,3\rangle, \langle 2,2\rangle, \langle 2,1\rangle, \langle 3,3\rangle\}$.

7. (a) $\neg((\neg A \wedge B) \vee C) \vee (A \wedge \neg D)$

$\Leftrightarrow$

$(\neg(\neg A \wedge B) \wedge \neg C) \vee (A \wedge \neg D)$

$\Leftrightarrow$

$((\neg\neg A \vee \neg B) \wedge \neg C) \vee (A \wedge \neg D)$

$\Leftrightarrow$

$((A \vee \neg B) \wedge \neg C) \vee (A \wedge \neg D)$

$\Leftrightarrow$

$(A \vee \neg B \vee (A \wedge \neg D)) \wedge (\neg C \vee (A \wedge \neg D))$

$\Leftrightarrow$

$(A \vee \neg B \vee A) \wedge (A \vee \neg B \vee \neg D) \wedge (\neg C \vee A) \wedge (\neg C \vee \neg D)$

$\Leftrightarrow$

$(A \vee \neg B) \wedge (A \vee \neg B \vee \neg D) \wedge (\neg C \vee A) \wedge (\neg C \vee \neg D)$

[$\Leftrightarrow$

$(A \vee \neg B) \wedge (\neg C \vee A) \wedge (\neg C \vee \neg D)$]

(b) $\exists x \forall y R(x,y) \to \forall x \exists z Q(x,z)$

$\Leftrightarrow$ { rename x in the conclusion }

$\exists x \forall y R(x,y) \to \forall w \exists z Q(w,z)$

$\Leftrightarrow$

$\forall x(\forall y R(x,y) \to \forall w \exists z Q(w,z))$

$\Leftrightarrow$

$\forall x \exists y(R(x,y) \to \forall w \exists z Q(w,z))$

$\Leftrightarrow$

$\forall x \exists y \forall w(R(x,y) \to \exists z Q(w,z))$

$\Leftrightarrow$

$\forall x \exists y \forall w \exists z(R(x,y) \to Q(w,z))$

Skolemize $\exists y$:

$\forall x \forall w \exists z(R(x,f(x)) \to Q(w,z))$

Skolemize $\exists z$:

$\forall x \forall w(R(x,f(x)) \to Q(w,g(x,w)))$

(c)

A	B	D	E	$A \wedge ((A \wedge B \wedge E) \to$	$\bot) \wedge$	$E \wedge (E \to$	$D) \wedge (B \to$	$A) \wedge ((D \wedge E) \to$	$B)$
T	T	T	T	T	T?	T	T		T
1	4	3	2	1	5	2	3		4

In step 1, we assign T to A, because it is a conjunct of the Horn sentence.
In step 2, we do the same for E.
In step 3, we observe that the premiss of conjunct $E \to D$ is true, so we assign T to D.
In step 4, we observe that the premiss of conjunct $(D \wedge E) \to B$ is true, so we assign T to B.
In step 5, we observe that the premiss of conjunct $(A \wedge B \wedge E) \to \bot$ is true, so we should assign T to $\bot$, which is impossible.
We conclude that the Horn sentence cannot be satisfied.

8.

1. $\forall x \exists y (P(x) \wedge Q(y))$	
2. $\exists y (P(a) \wedge Q(y))$	$\forall$ Elim: 1
3. $\boxed{b}$ $P(a) \wedge Q(b)$	
4. $\boxed{c}$	
5. $\exists y (P(c) \wedge Q(y))$	$\forall$ Elim: 1
6. $\boxed{d}$ $P(c) \wedge Q(d)$	
7. $P(c)$	$\wedge$ Elim: 6
8. $P(c)$	$\exists$ Elim: 5, 6–7
9. $Q(b)$	$\wedge$ Elim: 3
10. $P(c) \wedge Q(b)$	$\wedge$ Intro: 8, 9
11. $\forall x (P(x) \wedge Q(b))$	$\forall$ Intro: 4–10
12. $\exists y \forall x (P(x) \wedge Q(y))$	$\exists$ Intro: 11
13. $\exists y \forall x (P(x) \wedge Q(y))$	$\exists$ Elim: 2, 3–12

Other proofs are possible, using the $\neg$ Elimination rule. I think that this is the shortest proof.